Grids and Related Problems

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1 Colouring and Weights

Problems relating to grids appear frequently and can require techniques from all corners of combinatorics, including graph theory, induction, algorithmic ideas, pigeonhole/probabilistic method and the extremal principle/greedy. Each of these ideas appear in more than one problem in these notes. One technique especially useful in grid problems is introducing a colouring or, often equivalently, assigning weights to grid squares. We begin with the classic intro problem to colourings.

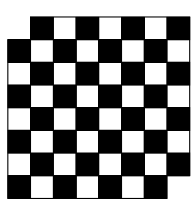
Example 1. *A pair of diagonally opposite corners are removed from an* 8×8 *grid. Can the resulting figure be tiled with dominos?*

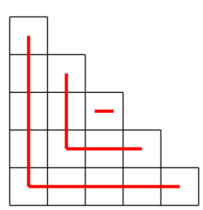
Solution. No it cannot. Colour the squares of the 8×8 grid black and white as in a chessboard. The two removed squares are both the same colour and thus the resulting figure has unequal numbers of black and white squares. Each domino covers exactly one black and one white square. This implies that any figure that can be tiled with dominos must have an equal number of black and white squares, implying this figure cannot. □

These types of problems are quite common – introduce a colouring so that when you throw down some shape, it intersects this colouring in a predictable way. This usually gives impossibility results i.e. that certain figures cannot be tiled with certain shapes. However, colourings are good for more than that! In the next example, we see that they can give tight *extremal* bounds.

Example 2. *Consider an* $n \times n$ *staircase, which consists of the squares on or below the main diagonal of an* $n \times n$ *grid. A path is a sequence of distinct squares, every two consecutive of which share an edge. What is the minimum number of paths that an* $n \times n$ *staircase can be partitioned into?*

Solution. We will show that $\lceil n/2 \rceil$ paths are the minimum necessary. This can be achieved using L-shaped paths between the first and last squares of the diagonal, between the second and second last squares, etc. Colour the squares of the staircase like a chessboard so that its diagonal is all black. Each path in the previous decomposition contains exactly one more black square than white square. Thus the staircase contains $\lceil n/2 \rceil$ more black squares than white squares. Since any path contains at most one more black square than white square, at least $\lceil n/2 \rceil$ paths are necessary. □





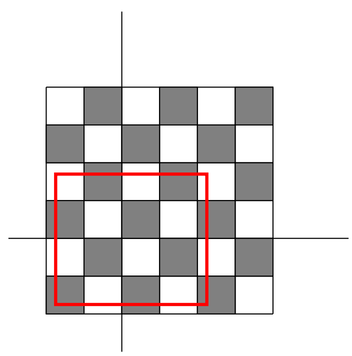
Sometimes more than two colours can be useful! In many of the problems at the end of these notes, three colours is the right number. In the next example, we use a variable number.

Example 3. *(ToT 2009) We place 2009 $n \times n$ cardboard pieces each covering exactly n^2 squares on an infinite chessboard. Prove that the number of cells of the chessboard which are covered by odd numbers of cardboard pieces is at least n^2.*

Solution. Let $(0, 0), (0, 1), \dots, (n-1, n-1)$ be labels for n^2 colours. Colour the square (x, y) with colour $(x \pmod n), y \pmod n)$ and note that every $n \times n$ grid contains exactly one copy of each colour. Each colour (a, b) is covered by exactly once by each of the 2009 cardboard pieces, implying that there is some square of that colour covered an odd number of times. Thus there is at least one square of each of the n^2 colours covered by an odd number of cardboard pieces, proving the result. □

The next example is a classical result with applications beyond grids and tiling. There are many alternative proofs including integrating various function or finding paths of rectangles. We give a simple colouring proof which illustrates that colouring can be useful even when rectangles are not grid-aligned.

Example 4. *The interior of a rectangle R is partitioned into rectangles with sides parallel to R such that each rectangle has at least one side which has an integer length. Prove that R has a side which has an integer length.*



Solution. Scale R and the rectangles inside of it up by a factor of two, place the origin $(0, 0)$ at the bottom left corner of R and its bottom and left sides along the x and y axes. Colour the squares of the plane like a chessboard. We now make two observations:

- **Claim 1.** Any axis-parallel rectangle R' with a side with an even-integer side length contains an equal area of black and white.
  Proof. Suppose w.l.o.g. that the horizontal side of R' has even length $2m$ and lower left corner (x, y). Note that (a, b) and $(a + 2m, b)$ have the same colour for all (a, b). Thus R' contains the same amount of each colour as the rectangle R'' with lower left corner $(0, y)$ congruent to R'. Reflecting R'' about its axis of symmetry $x = m$ brings points to points of opposite colours. This implies R'' contains equal amounts of black and white. □

- **Claim 2.** Any axis-parallel rectangle with lower left corner $(0, 0)$ that contains equal amounts of black and white has an even-integer side length.

Proof. Assume for contradiction that it does not. Let its upper right corner be (x, y). Let $0 < a, b < 2$ be the remainders when its horizontal and vertical side lengths are divided by 2. Applying Claim 1 twice yields that the rectangle R' with lower left corner $(0, 0)$ and upper right corner (a, b) must contain equal amounts of black and white. Suppose the lattice square with lower left corner $(0, 0)$ is black. Then if $\min(a, b) \le 1$, then R' clearly contains more black than white. Otherwise, it contains $(a-1)(b-1)+1$ black area and $a+b-2$ white area. The difference between these is $(2-a)(2-b) > 0$, which is a contradiction. □

Now each rectangle in the partition of R contains equal amount of black and white by Claim 1. By Claim 2 this implies that R has an even-integer side length after being scaled by a factor of two. Thus R originally had an integer side length. □

We now give a simple corollary of Example 4. When k is prime, this result follows from the fact that the area k of a $1 \times k$ must divide the area mn of the grid, which implies that k divides one of m or n. However, when k is composite, area arguments do not suffice but the result is still true. We also give two other proofs using a colouring and complex number weights. As is often the case, introducing complex numbers yields an algebraically slick way of stating what is morally equivalent to a colouring argument. Even though they often aren't doing something you couldn't do with a colouring argument, they can make a problem easier to reason about.

Example 5. *Suppose an $m \times n$ grid can be tiled with $1 \times k$ dominos, then k divides one of m or n.*

Proof 1. Scale both axes by a factor of $1/k$. Each $1 \times k$ domino now has a side length of 1. Thus the scaled $m \times n$ grid must have an integer side-length by Example 4, implying the result. □

Proof 2. Colour the $m \times n$ grid with k colours $\{0, 1, \dots, k-1\}$ such that the square (x, y) has the colour $x + y \pmod k$. Each $1 \times k$ and $k \times 1$ domino covers one square of each colour so it suffices to show that any grid with m, n not divisible by k does not have an equal number of each colour. If m or n is greater than k, we can remove k rows or k columns since any k rows or columns contain an equal amount of each colour. Thus we may assume that $0 < m, n < k$. Assume w.l.o.g. that $m \le n$. Consider the diagonal containing the bottom right corner of the grid. All m squares on this diagonal have the same colour. Thus a fraction of $m/mn = 1/n > 1/k$ of the squares have the same colour, making it impossible for the grid to contain equal amounts of each colour. □

Proof 3. Let ω be the kth root of unity $\omega = e^{2\pi i/k}$. Place the weight ω^{i+j-2} in cell (i, j). Note that in any $1 \times k$ or $k \times 1$ subgrid, there is exactly one square with each of the weights $1, \omega, \dots, \omega^{k-1}$. Thus the sum of the weights in any domino is zero, implying that if the grid can be tiled then the sum of its weights is zero. Now note that the total sum of weights in the grid is

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \omega^{i+j} = \left(\sum_{i=0}^{m-1} \omega^i \right) \left(\sum_{j=0}^{n-1} \omega^j \right) = \frac{(\omega^m - 1)(\omega^n - 1)}{(\omega - 1)^2}$$

which implies that either $\omega^m = 1$ or $\omega^n = 1$. Thus k divides one of m or n. □

One beauty of introducing weights is that they often reduce various conditions to linear equations over the complex numbers. Bijections and double counting arguments can boil down to manipulating a linear system with weights. We will illustrate this with the next problem from IMO 2016. Many grid problems ultimately come down to a linear system of equations, sometimes over

$\mathbb{F}_2$. Many slick parity arguments can be thought of as manipulating a system of equations! We will demonstrate this later.

Example 6. *(IMO 2016) Find all integers n for which each cell of $n \times n$ table can be filled with one of the letters I, M and O in such a way that:*

- *in each row and each column, one third of the entries are I, one third are M and one third are O; and*
- *in any diagonal, if the number of entries on the diagonal is a multiple of three, then one third of the entries are I, one third are M and one third are O.*

Solution. The answer is all n divisible by 9. These n can be achieved by stacking copies of the following 9×9 table:

I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M
I	I	I	M	M	M	O	O	O
M	M	M	O	O	O	I	I	I
O	O	O	I	I	I	M	M	M

We will use weights to show that 9 divides n by simply manipulating equations. Let $\omega = e^{2\pi i/3}$ be a third root of unity. Given such a table, write 1 for an I, ω for an M and ω^2 for an O. Given three real numbers a, b, c, it holds that $a + b\omega + c\omega^2 = 0$ if and only if $a = b = c$. Therefore the sum of the weights in every column, row and every third diagonal is zero. Note that since each row consists of a multiple of three entries, we have that 3 divides n.

We have a lot of equations (in total $8n/3$). Searching directly for an informative linear combination of these equations could get really messy, so it seems natural to start with the simplest linear combinations possible. A first idea is to sum them all and observe that the sum of all entries is zero. This isn't very helpful and doesn't even require all of the equations to be true. So we try the next most coarse approach, grouping every third equation together. Manipulating these groups seems as though it may allow us to use all of the given equations. Formally, we group variables by letting a_{ij} be the sum of the weights in all cells (i', j') with $i \equiv i' \pmod 3$ and $j \equiv j' \pmod 3$ for each $1 \le i, j \le 3$. Our equations now imply that

$$a_{11} + a_{12} + a_{13} = 0 \tag{1}$$
$$a_{21} + a_{22} + a_{23} = 0 \tag{2}$$
$$a_{31} + a_{32} + a_{33} = 0 \tag{3}$$
$$a_{11} + a_{21} + a_{31} = 0 \tag{4}$$
$$a_{12} + a_{22} + a_{32} = 0 \tag{5}$$
$$a_{13} + a_{23} + a_{33} = 0 \tag{6}$$
$$a_{11} + a_{22} + a_{33} = 0 \tag{7}$$
$$a_{31} + a_{22} + a_{13} = 0 \tag{8}$$

Let's try to isolate a variable if we can. Observe that

$$0 = (1) + (3) + (4) + (6) - 2 \cdot (7) - 2 \cdot (8) - (2) - (5) = -6a_{22}$$

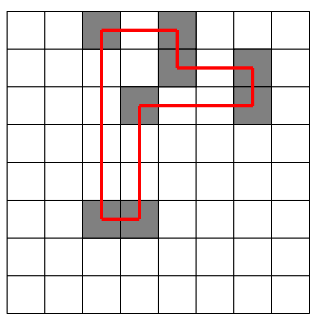
So $a_{22} = 0$ but this implies that the squares with indices congruent to 2 modulo 3 must contain an equal number of I's, M's and O's. So $n^2/9$ is divisible by 3, implying 9 divides n. □

After realizing that the key is to show that these squares contain an equal number of I's, M's and O's, we can also show this by double counting.

2 Graph Theory and Grids

Many ideas in the previous section can be thought of graph-theoretically. A domino tiling is a perfect matching of the squares of a grid where two squares are adjacent if they share an edge. The first example realizes that a chessboard is bipartite with the two colours corresponding to the two parts of the graph and that a perfect matching in a bipartite graph is possible only if the two parts have the same size. The next example requires a different bipartite graph representation of grids where squares correspond to edges between their row and column. Here we essentially use the fact that any graph on n vertices with at least n edges contains a cycle.

Example 7. *In an $n \times n$ grid, at least $2n$ squares are marked. Prove that there is a sequence $P_1, P_2, \dots, P_k$ of centeres of marked squares such that the segments P_iP_{i+1} alternate between horizontal and vertical for all $1 \leq i \leq k$ where $P_{k+1} = P_1$.*



Solution. Consider the bipartite graph G with $2n$ vertices $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n$ such that u_iv_j is an edge if and only if the square (i, j) is marked. If G does not contain a cycle, it must be a forest and contains $2n - c$ edges where $c \geq 1$ is the number of connected components of the G. However this contradicts the fact that G contains at least $2n$ edges. Thus G contains a cycle. The edges of this cycle correspond to cells that alternate being in the same column and being in the same row, proving the desired result. □

Depending on the specific problem, other graph representations of grids may be useful. In the next problem, we use the fact that any undirected graph where every vertex has degree 2 is a disjoint union of cycles. We will use the geometry of the grid to infer a structural property of its graph representation – namely that all cycles in the graph have even length.

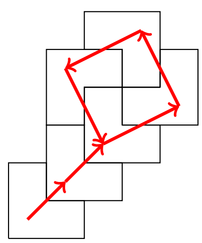
Example 8. *(Gabriel Carroll) A company wants to build a 2001×2001 building with doors connecting pairs of adjacent rooms. Two rooms are adjacent if they share an edge. Is it possible for every room to have exactly 2 doors?*

Solution. No it is not possible. Consider the graph G with 2001^2 vertices, each corresponding to room, such that two rooms are connected by an edge if there is a door between them. If every room has exactly two doors, then every vertex in G has degree two, implying that G is a disjoint union of cycles. Now colour the grid as in a chessboard. Adjacent rooms have different colours and therefore any cycle alternates colours, implying it is even in length. Therefore G must have an even number of vertices if it is a disjoint union of cycles, which is a contradiction. □

Note that the above solution essentially observes that G is bipartite with the two parts given by the colours in a chessboard colouring, as in Example 1. The fact that any finite graph in which every vertex has degree two is a disjoint union of cycles can be proven by strong induction. Begin at an arbitrary vertex and take an edge from it. Then take an edge from the new vertex other than the one just taken to get there. Continue in this way until you see a vertex for the second time. This vertex must be the vertex you started at, otherwise it would have degree at least three. Furthermore, we have exhausted the degrees of the vertices we have seen with this cycle. Applying the induction hypothesis to the rest of the graph yields the result.

The next example uses deeper properties of the geometry of the grid to infer something about a related graph. It also illustrates the usefulness of considering grid vertices and edges as tools in problems about grid squares.

Example 9. *(St. Petersburg 2000) On an infinite checkerboard are placed* 111 *non-overlapping corners, L-shaped figures made of* 3 *unit squares. Suppose that for any corner, the* 2×2 *square containing it is entirely covered by the corners. Prove that one can remove each number between* 1 *and* 110 *of the corners so that the property will be preserved.*



Solution. Consider the *directed* graph G with 111 vertices, each corresponding to an L-shaped figure. Draw a directed edge from each L-shape u to the L-shape v such that a square of v occupies the missing square of the 2×2 box containing u. As given, every vertex in G has out-degree exactly one. We are looking for a subset with between 1 and 110 vertices of G such that every vertex has out-degree exactly one on the subset.

If there are two vertices that both point to each other, they must together form a 2×3 or 3×2 box, that we can remove. If there is a vertex with in-degree zero, we can just remove that vertex. Now note that if there is a vertex with in-degree other than one, there must be a vertex with in-degree zero since the sum of the in-degrees equals the sum of the out-degrees. Thus we can assume that every vertex of G has in-degree one and no two vertices point to each other. By a directed analogue of the argument above, G must be the disjoint union of cycles, each of which has length at least three. Now consider the centers of the 2×2 boxes containing any two L-shapes u and v with $u \to v$. If u and v do not point to each other, some casework shows that the x coordinates of these centers have different parity. This implies that any cycle in G is even and thus G contains an even number of vertices. This is a contradiction. □

Note that the proof above actually shows the stronger result that we can remove exactly one corner while preserving the property.

3 Some Other Techniques

There are many other ideas that appear in grid problems. Some of these include:

- Algorithms to show existence
- Extremal arguments
- Induction
- Parity arguments and double-counting

The next example finds a desired substructure algorithmically. As with many algorithmic existence proofs, it can be written as an induction, thought of as generalizing the problem (e.g. here to removing a single row from $m \times n$ grids) or as dynamic programming. It also illustrates the power of making use of parity, which is the reason behind the entire result. If zeros were allowed in the grid as well, it would be far from true.

Example 10. *(David Arthur) Numbers* 1 *and* -1 *are written in the cells of a board* 1500×1500*. It is known that the sum of all the numbers in the board is positive. Show that one can select* 1000 *rows and* 1000 *columns such that the sum of numbers written in their intersection cells is at least* 1000*.*

Solution. First we prove the following general lemma.

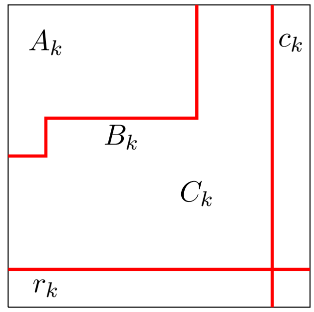
Lemma 1. *The numbers* 1 *and* -1 *are written in a* $m \times n$ *grid with sum* S*. If* n *is odd, we can delete a row so that the remaining grid has sum at least* $\min(S+1, m-1)$*. If* n *is even, then we can delete a row so that the remaining grid has sum at least* $\min(S, m-1)$

Proof. If there is a row with sum at most 0, deleting this yields a sum of at least S. If n is odd, then every row has odd sum. Thus if there is a row with sum at most 0, it must have sum at most -1. Deleting this row yields a sum of at least $S + 1$. If there is no row with sum at most 0, then deleting any row yields a sum of at least $m - 1$. This proves the lemma. □

Note that the sum of the grid must initially be at least 2 since it is even. Now delete one row to yield a 1499×1500 grid with sum at least 2. Delete 499 columns to yield a 1499×1001 grid with sum at least 501. Delete 499 rows to yield a 1000×1001 grid with sum at least 1000. Now delete one column to yield a 1000×1000 grid with sum at least 1000. □

The next problem uses a very tricky extremal argument, the basic idea behind which is geometrically intuitive. The key is to use the fact that any set of $n - 1$ consecutive values geometrically separates the squares corresponding to the values above and below it in the grid. However, $n - 1$ squares seems like too few to separate sufficiently large sets of squares. Making this argument rigorous ends up requiring some clever ideas.

Example 11. *(ISL 1988) The numbers* $1, 2, \ldots, n^2$ *are written in an* $n \times n$ *square grid. Prove that there is some pair of adjacent squares with difference at least* n*.*



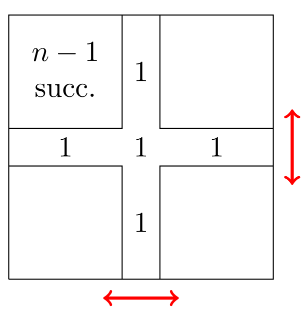
Solution. Assume for contradiction that every pair of adjacent squares has difference at most $n-1$. For each $k = 1, 2, \dots, n^2 - n$, define

- A_k to be the set of squares with values $1, 2, \dots, k$
- B_k to be the set of squares with values $k+1, \dots, k+n-1$
- C_k to be the set of squares with values $k+n, \dots, n^2$

No square in A_k can be adjacent to a square in C_k i.e. B_k separates A_k from C_k. Since $|B_k| = n-1$, there is some row r_k and some column c_k that does not intersect B_k. Since r_k and c_k intersect in a square, it either holds that $r_k, c_k \subseteq A_k$ or $r_k, c_k \subseteq C_k$ for each k. Note that $r_1, c_1 \subseteq C_1$ and $r_{n^2-n}, c_{n^2-n} \subseteq A_{n^2-n}$ since $|A_1|, |C_{n^2-n}| = 1 < n$. Now let $t > 1$ be the first t such that $r_t, c_t \subseteq A_t$. Since $r_{t-1}, c_{t-1} \subseteq C_{t-1}$, it follows that A_t and C_{t-1} must intersect in the two squares $r_{t-1} \cap c_t$ and $r_t \cap c_{t-1}$. However, A_t and C_{t-1} intersect in exactly one square, which is a contradiction. □

The next example is an induction with a non-standard reduction to smaller cases. In this problem, we observe that two good objects can be composed in a natural way to yield another good object (e.g. in this case taking the element-wise product of two successful arrangements). So given a good object A, we face the question: how do we generate another object B to compose with A in a useful way? Here, we generate B by applying a natural symmetry to A. One very hard problem at the end of these notes uses a similar idea!

Example 12. *(Russia 1998) Let $n \geq 2$ be a positive integer. Each square of a $(2^n - 1) \times (2^n - 1)$ board contains either 1 or -1. Such an arrangement is called successful if each number is the product of its neighbors. Find the number of successful arrangements.*



Solution. We prove by induction on n that the only possible arrangement contains only 1's for $n \geq 2$. We first deal with the base case $n = 2$. Let a_{ij} where $1 \leq i, j \leq 3$ be any successful arrangement. First observe that $a_{22} = a_{21}a_{23}a_{12}a_{32} = a_{22}^4 a_{11}^2 a_{13}^2 a_{31}^2 a_{33}^2 = 1$. Also $a_{11}a_{13} = a_{12}^2 a_{21} a_{23} =$

$(a_{22}a_{11}a_{31})(a_{22}a_{13}a_{33})$ which implies that $a_{31}a_{33} = 1$. Therefore $a_{32} = a_{22}a_{31}a_{33} = 1$. Similarly $a_{12} = a_{32} = a_{23} = a_{21} = 1$. This implies that all of the entries of a_{ij} are 1, proving the base case.

Assume for contradiction that there is some arrangement a'_{ij} for $1 \le i, j \le 2^n - 1$ satisfying the given conditions with some $a'_{ij} = -1$. We will show that there is some arrangement a_{ij} such that the 2^{n-1}th row and column both consist of all 1's with some $a'_{ij} = -1$. Note that the element-wise product of any two successful arrangements is also successful. If a'_{ij} is not symmetric about its vertical axis of symmetry, multiply it by its reflection in this axis. Since it is not symmetric, it must still contain a -1. Do the same for the horizontal axis of symmetry of the new arrangement. Call the resulting arrangement a_{ij}, which is both horizontally and vertically symmetric, and contains a -1. Let $k = 2^{n-1}$ and consider the elements in the kth column of a_{ij}. It follows that $a_{1,k} = a_{1,k-1}a_{1,k+1}a_{2,k} = a_{2,k}$ since $a_{1,k-1} = a_{1,k+1}$ by symmetry. Similarly, for all $1 \le t \le k-1$, we have that $a_{t+1,k} = a_{t,k}a_{t+2,k}$ and $a_{2^n-1,k} = a_{2^n-2,k}$. If $a_{1,k} = a_{2,k} = 1$, then these equations imply that $a_{t,k} = 1$ for all t. If $a_{1,k} = a_{2,k} = -1$, then $a_{3,k} = 1$ and in general

$$a_{t,k} = \begin{cases} 1 & \text{if } t \equiv 1, 2 \pmod 3 \\ -1 & \text{if } t \equiv 0 \pmod 3 \end{cases}$$

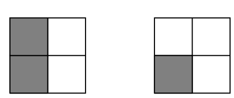
However, $2^n - 1 \not\equiv 2 \pmod 3$ which implies that the sequence $a_{1,k}, \dots, a_{2^n-1,k}$ is not symmetric if it takes this form. Therefore it must follow that $a_{t,k} = 1$ for all t. By the same reasoning the kth row must also consist of all 1's, which proves the claim.

Since a_{ij} is horizontally and vertically symmetric, any of its four $(2^{n-1} - 1) \times (2^{n-1} - 1)$ corner subgrids contains a -1. Furthermore, since the 2^{n-1}th row and column consist of all 1's, these four subgrids must each be successful for $n - 1$. The induction hypothesis yields a contradiction. □

If we let $a_{ij} = 1$ if there is a -1 in square (i, j) and $a_{ij} = 0$ otherwise, then every a_{ij} is equal to the sum of its neighboring values modulo 2. In the above proof, we show that any solution to these equations over $\mathbb{F}_2$ must be all zeros. In linear algebra terms, that the linear system is invertible. However, to actually show this we need to use the geometry of the grid as in the above solution.

The next example gives several disallowed local structures which together yield a contradiction. A common trick in these problems is to look locally for a "nice" characterization e.g. some count takes on specific values. This often leads to a double counting argument as in the example below.

Example 13. *(Russia 2017) Each cell of 100×100 table is coloured black or white. Every cell sharing an edge with the boundary of the table is coloured black. Suppose that in every 2×2 square there are cells of both colors. Prove that there exists a 2×2 square that is coloured like a 2×2 subgrid of a chessboard.*



Solution. Assume for contradiction that there is no 2×2 square coloured as in a chessboard. Call an edge between two squares *good* if it separates two squares of opposite colours. The key observation is that the allowed 2×2 patterns – those with three squares of the same colour or two pairs of adjacent squares of the same colour – each contain exactly two good edges. This is the nice characterization of local structure that we need. There are 99^2 total 2×2 subgrids, each of which contains two good edges. Since no good edges lie in the first or last row or column, we have counted each good edge in exactly two 2×2 subgrids. Therefore there are exactly 99^2 good edges.

However, each row or column begins and ends with the colour black, implying it changes colour at an even number of good edges. Therefore the total number of good edges is even, which is a contradiction. □

Problems

We adopt the David Arthur style of dividing the following problems into several difficulty classes A, B, C and D. They are (hopefully) roughly ordered by difficulty.

A1. 100 queens are placed on a 100×100 chessboard so that no two attack each other. Prove that each of the four 50×50 corners of the board contains at least one queen.

A2. Is it possible to tile 2003×2003 board by 1×2 dominoes placed horizontally and 1×3 rectangles placed vertically?

A3. James and Alex play a game on a $n \times n$ chessboard. At the beginning, all squares are white apart from one black corner square containing a rook. Players take turns to move the rook to a white square and recolour the square black. The player who can not move loses. James goes first. Who has a winning strategy?

A4. A square has been removed from a $2^n \times 2^n$ grid. Prove that the remaining figure can be tiled with L-trominos.

A5. Some of the fields on a rectangular $m \times n$ board ($m, n \geq 2$) are painted black, while the remaining fields are white. There is a frog sitting outside the board. At a certain moment the frog jumps onto one of the fields on the edge of the board, and then makes a sequence of jumps. In every jump, it jumps from a field to its neighbouring field (two fields are neighbouring if they have a side in common). Every time the frog jumps to a field, the colour of that field changes either from black to white or vice-versa. Is there a path the frog could take so that, when it leaves the board by jumping from a field on the edge, all the fields are black?

A6. Each edge of an $m \times n$ rectangular grid is oriented with an arrow such that (a) the border is oriented clockwise, and (b) each interior vertex has two arrows coming out of it, and two arrows going into it. Prove that there is at least one square whose edges are oriented clockwise.

A7. Prove that it is not possible colour the squares of a 11×11 grid using three colours, such that no four squares whose centres form the vertices of a rectangle with sides parallel to the sides of the grid, have the same colour.

A8. On a $(4n + 2) \times (4n + 2)$ square grid, a turtle can move between squares sharing a side.The turtle begins in a corner square of the grid and enters each square exactly once, ending in the square where she started. In terms of n, what is the largest positive integer k such that there must be a row or column that the turtle has entered at least k distinct times?

A9. On an $n \times n$ table real numbers are put in the unit squares such that no two rows are identically filled. Prove that one can remove a column of the table such that the new table has no two rows identically filled.

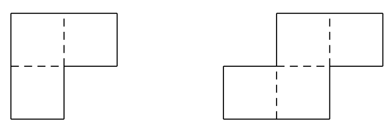
A10. In an $n \times n$ array, each of the numbers $1, 2, \ldots, n$ appears exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.

A11. There are k rooks on a 10×10 chessboard. We mark all the squares that at least one rook can capture (we consider the square where the rook stands as captured by the rook). What is the maximum value of k so that the following holds for some arrangement of k rooks: after removing any rook from the chessboard, there is at least one marked square not captured by any of the remaining rooks.

A12. Can a 5×7 checkerboard be covered by L's, not crossing its borders, in several layers so that each square of the board is covered by the same number of L's?

A13. A maze is an 8×8 board with some adjacent squares separated by walls, so that any two squares can be connected by a path not meeting any wall. Given a command LEFT, RIGHT, UP, DOWN, a pawn makes a step in the corresponding direction unless it encounters a wall or an edge of the chessboard. Eddy writes a program consisting of a finite sequence of commands and gives it to the Calvin, who then constructs a maze and places the pawn on one of the squares. Can Eddy write a program which guarantees the pawn will visit every square despite the Calvin's efforts?

B1. Consider a $(2m-1) \times (2n-1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. The region is to be tiled using tiles of the two types shown:



The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping. What is the minimum number of tiles required to tile the region?

B2. A number of robots are placed on the squares of a finite, rectangular grid of squares. A square can hold any number of robots. Every edge of each square of the grid is classified as either passable or impassable. All edges on the boundary of the grid are impassable. You can give any of the commands up, down, left, or right. All of the robots then simultaneously try to move in the specified direction. If the edge adjacent to a robot in that direction is passable, the robot moves across the edge and into the next square. Otherwise, the robot remains on its current square. You can then give another command of up, down, left, or right, then another, for as long as you want. Suppose that for any individual robot, and any square on the grid, there is a finite sequence of commands that will move that robot to that square. Prove that you can also give a finite sequence of commands such that all of the robots end up on the same square at the same time.

B3. A 6×6 rectangle is tiled by 2×1 dominoes. Prove that it has always at least one fault-line, i.e., a line cutting the rectangle without cutting any domino.

B4. A chessboard is tiled with 32 dominoes. Each domino covers two adjacent squares, a white and a black square. Show that the number of horizontal dominoes with the white square on the left of the black square equals the number of horizontal dominoes with the white square on the right of the black square.

B5. Every vertex of the unit squares on an $m \times n$ grid is coloured either blue, green, or red, such that all the vertices on the boundary of the board are coloured red. We say that a unit square on the board is properly coloured if exactly one pair of adjacent vertices of the square are the same colour. Show that the number of properly coloured squares is even.

B6. On an $n \times n$ chart, where $n \geq 4$, stand $+$ signs in the cells of the main diagonal and $-$ signs in all the other cells. You can change all the signs in one row or in one column, from $-$ to $+$ or from $+$ to $-$. Prove that you will always have n or more $+$ signs after finitely many operations.

B7. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively. Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove that no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

B8. Several fleas sit on the squares of a 10×10 chessboard (at most one flea per square). Every minute, all fleas simultaneously jump to adjacent squares. Each flea begins jumping in one of four directions (up, down, left, right), and keeps jumping in this direction while it is possible; otherwise, it reverses direction on the opposite. It happened that during one hour, no two fleas ever occupied the same square. Find the maximal possible number of fleas on the board.

B9. A square is divided into congruent rectangles with sides of integer lengths. A rectangle is important if it has at least one point in common with a given diagonal of the square. Prove that this diagonal bisects the total area of the important rectangles.

B10. A rectangle is divided into 2×1 and 1×2 dominoes. In each domino, a diagonal is drawn, and no two diagonals have common endpoints. Prove that exactly two corners of the rectangle are endpoints of these diagonals.

B11. On the infinite chessboard several rectangular pieces are placed whose sides run along the grid lines. Each two have no squares in common, and each consists of an odd number of squares. Prove that these pieces can be painted in four colours such that two pieces painted in the same colour do not share any boundary points.

B12. On an $n \times n$ board, there are n^2 squares, $n-1$ of which are infected. Each second, any square that is adjacent to at least two infected squares becomes infected. Show that at least one square always remains uninfected.

B13. On a 55×55 square grid, 500 unit squares were cut out as well as 400 L-shaped pieces consisting of 3 unit squares (each piece can be oriented in any way). Prove that at least two of the cut out pieces bordered each other before they were cut out.

B14. 2500 chess kings have to be placed on a 100×100 chessboard so that (i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex); (ii) each row and each column contains exactly 25 kings. Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

B15. In a rectangular array of nonnegative real numbers with m rows and n columns, each row and each column contains at least one positive element. Moreover, if a row and a column intersect at a positive element, then the sums of their elements are the same. Prove that $m = n$.

B16. Let n be a positive integer. Denote by S_n the set of points (x, y) with integer coordinates such that

$$|x| + \left|y + \frac{1}{2}\right| < n.$$

A path is a sequence of distinct points $(x_1, y_1), (x_2, y_2), \ldots, (x_\ell, y_\ell)$ in S_n such that, for $i = 2, \ldots, \ell$, the distance between (x_i, y_i) and (x_{i-1}, y_{i-1}) is 1 (in other words, the points (x_i, y_i) and (x_{i-1}, y_{i-1}) are neighbors in the lattice of points with integer coordinates). Prove that the points in S_n cannot be partitioned into fewer than n paths (a partition of S_n into m paths is a set $\mathcal{P}$ of m nonempty paths such that each point in S_n appears in exactly one of the m paths in $\mathcal{P}$).

B17. A plane is coloured into black and white squares in a chessboard pattern. Then, all the white squares are coloured red and blue such that any two initially white squares that share a corner are different colours. (One is red and the other is blue.) Let ℓ be a line not parallel to the sides of any squares. For every line segment I that is parallel to ℓ, we can count the difference between the length of its red and its blue areas. Prove that for every such line ℓ there exists a number C that exceeds all those differences that we can calculate.

B18. An *arrowgram* is a finite rectangular grid with an arrow drawn in each square such that:

- Each arrow points to an adjacent square in one of the eight compass directions (and does not point off the edge of the grid), and
- No two arrows point to the same square.

Two arrowgrams A and B are said to be similar if they are on equally sized grids, and if for every square, the corresponding arrows in A and B either point in the same direction or in opposite directions. For what integers N does there exist an arrowgram that is equivalent to exactly N other arrowgrams?

C1. A solitaire game is played on an $m \times n$ rectangular board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner square, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

C2. There is an $n \times n$ grid on a computer. Each of its n^2 squares displays an integer from 0 to k. For each of the n rows and each of the n columns, there is also a button that, if pressed, will increase every number in that row or column by 1. If a number ever reaches k, it immediately changes to 0. Initially, every square displayed 0, but then a number of buttons were pressed.

Show that after at most kn more button presses, it is possible to change every number back to 0 again.

C3. The 52 cards in a standard deck are placed in a 13×4 array. If every two adjacent cards, vertically or horizontally, have either the same suit or the same value, prove that all 13 cards of the same suit are in the same row.

C4. Each cell of a 1000×1000 table contains 0 or 1. Prove that one can either cut out 990 rows so that at least one 1 remains in each column, or cut out 990 columns so that at least one 0 remains in each row.

C5. There are some counters in some cells of 100×100 board. Call a cell nice if there are an even number of counters in adjacent cells. Can exactly one cell be nice?

C6. A white plane is partitioned onto cells (in a usual way). A finite number of cells are coloured black. Each black cell has an even (0, 2 or 4) adjacent (by the side) white cells. Prove that one may colour each white cell in green or red such that every black cell will have equal number of red and green adjacent cells.

C7. An $m \times n$ rectangular grid is covered by dominoes. Prove that the vertices of the grid can be coloured using three colours so that any two vertices a distance 1 apart are colored with different colours if and only if their segment lies on the boundary of a domino.

C8. All points in a 100×100 array are colored in one of four colors red, green, blue or yellow in such a way that there are 25 points of each color in each row and in any column. Prove that there are two rows and two columns such that their four intersection points are all in different colors.

C9. Given positive integers m and $n \geq m$, determine the largest number of dominoes (1×2 or 2×1 rectangles) that can be placed on a rectangular board with m rows and $2n$ columns consisting of cells (1×1 squares) so that: (i) each domino covers exactly two adjacent cells of the board; (ii) no two dominoes overlap; (iii) no two form a 2×2 square; and (iv) the bottom row of the board is completely covered by n dominoes.

C10. On an infinite chessboard, a solitaire game is played as follows: at the start, we have n^2 pieces occupying a square of side n. The only allowed move is to jump over an occupied square to an unoccupied one, and the piece which has been jumped over is removed. For which n can the game end with only one piece remaining on the board?

C11. In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominoes are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid. It is permitted to slide a domino towards the empty square, thus exposing another square. Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

D1. A $2^n \times n$ matrix of 1's and -1's is such that its 2^n rows are pairwise distinct. An arbitrary subset of the entries of the matrix are changed to 0. Prove that there is a nonempty subset of the rows of the altered matrix that sum to the zero vector.

D2. A frog named James hops around the squares of an $n \times n$ grid always to an adjacent square so that he visits each square exactly once and ends where he starts. Prove that there are two adjacent squares such that if we cut the cycle James takes at these squares, the number of squares in the two resulting pieces are each at least $n^2/4$.

D3. A frog is placed on each cell of a $n \times n$ square inside an infinite chessboard (so initially there are a total of $n \times n$ frogs). Each move consists of a frog A jumping over a frog B adjacent to it with A landing in the next cell and B disappearing (adjacent means two cells sharing a side). Prove that at least $\left[\frac{n^2}{3}\right]$ moves are needed to reach a configuration where no more moves are possible.

D4. A 2010×2010 board is divided into corner-shaped figures of three cells. Prove that it is possible to mark one cell in each figure such that each row and each column will have the same number of marked cells.

D5. Let $n \geq 3$ be an odd integer. Amy has coloured the squares in an $n \times n$ grid white and black. We will call a sequence of squares $S_1, S_2, \ldots, S_m$ a "path" if all these squares are the same colour, if S_i and S_{i+1} share an edge for all $i \in \{1, 2, \ldots, m-1\}$, and if no other squares in the sequence share an edge. Prove that if both the white squares and black squares form a single path, then one of these paths must begin or end at the center of the grid.

D6. Given natural numbers a and b, such that $a < b < 2a$. Some cells on a graph are colored such that in every rectangle with dimensions $a \times b$ or $b \times a$, at least one cell is colored. For which greatest α can you say that for every natural number N you can find a square $N \times N$ in which at least $\alpha \cdot N^2$ cells are colored?

D7. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contain two marked cells.